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Ballistic Behavior in a 1D Weakly Self-Avoiding Walk with Decaying Energy Penalty

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We consider a weakly self-avoiding walk in one dimension in which the penalty for visiting a site twice decays as $\exp[-\beta|t-s|^{-p}]$, where t and s are the times at which the common site is visited and p is a parameter. We prove that if $p<1$ and β is sufficiently large, then the walk behaves ballistically, i.e., the distance to the end of the walk grows linearly with the number of steps in the walk. We also give a heuristic argument that if $p>3/2$, then the walk should have diffusive behavior. The proof and the heuristic argument make use of a real-space renormalization group transformation.

KEY WORDS: Weakly self-avoiding walk; ballistic.

1. INTRODUCTION

The weakly self-avoiding walks which we will study are nearest-neighbor walks on a lattice which are allowed to self-intersect but are penalized when they do so. We denote the number of steps in the walk by N. A walk $\omega(t)$ is a function from $\{0, 1, 2,..., N\}$ into the lattice Z^d with $|\omega(t)-\omega(t-1)|=1$ for $t=1, 2,..., N$ and $\omega(0)=0$. For weakly self-avoiding walks one usually defines an energy function

$$H(\omega)=\sum_{1\leqslant s<t\leqslant N} 1(\omega(s)=\omega(t)) \tag{1}$$

which counts the number of self-intersections in the walk [$1(\omega(s)=\omega(t))$ is 1 if $\omega(s)=\omega(t)$ and is 0 otherwise]. We define a probability measure on the set of walks by

$$\text{Prob}(\omega)=\exp[-\beta H(\omega)]/Z \tag{2}$$

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where the normalization factor Z is defined by the requirement that this is a probability measure. (Throughout this paper the energy functions and the corresponding probability measures depend on N, the number of steps in the walk, but we will not make this dependence explicit.) We will always take the parameter β to be positive. For self-avoiding walks ($\beta = \infty$) one only allows nearest-neighbor walks that never visit the same site more than once. Each such walk of length N is then given the same probability.

There are many properties of these walks that are worth studying, but in this paper we will only be concerned with how the average distance from the origin to the end of the walk grows with the number of steps in the walk. It is expected that it grows as a power, i.e.,

$$\langle \omega(N)^2 \rangle \sim cN^{2\nu}$$

In five or more dimensions, Brydges and Spencer proved that $\nu = 1/2$ if β is sufficiently small.[4] Slade then proved that $\nu = 1/2$ for the self-avoiding walk if the dimension is large enough.[11] Finally, Hara and Slade proved that $\nu = 1/2$ for the self-avoiding walk if the dimension is at least five[7,8] (their proof also works for the weakly self-avoiding walk in five and more dimensions). In four dimensions, it is believed that $\nu = 1/2$ with logarithmic corrections. This has been proved in a hierarchical model,[2] and results have been obtained for the Green's function in a continuum model which is obtained by introducing a short-distance cutoff in the field theory representation of the Edwards model.[9]

In one dimension the self-avoiding walk is trivial and $\nu = 1$. We refer to this as ballistic behavior since it means that the distance traveled is proportional to the number of steps. Bolthausen[1] proved that $\nu = 1$ for the weakly self-avoiding walk if β is sufficiently small, and Zoladek[12] proved it for all $\beta > 0$ with a different definition of the energy function. Greven and den Hollander[6] showed that not only is $\nu = 1$, but the walk has a definite speed for any $\beta > 0$. In two and three dimensions there are no rigorous results on ν. A summary of the nonrigorous work as well as a more detailed discussion of the rigorous results mentioned here may be found in ref. 10.

In this paper we will consider a weakly self-avoiding walk in which the penalty for visiting a site twice decreases as the number of steps between the two visits increases. More precisely, we take the energy function to be

$$H(\omega) = \sum_{1 \leqslant s < t \leqslant N} \frac{1}{|t-s|^p} \, 1(\omega(s) = \omega(t)) \tag{3}$$

where p is a parameter. The probability measure on the set of walks of length N is defined as before. The larger the parameter p is, the faster the penalty for self-intersection falls off with the number of steps between the two visits. Thus p acts similarly to the dimension. (The larger the dimension is, the less likely the walk is to intersect itself, and so the penalty for self-intersection is less important.)

This model was studied by Caracciolo *et al.* for $2 \leqslant d \leqslant 4$.[5] They derived a variational mean-field approximation for the exponent ν and estimated it by Monte Carlo simulations in two dimensions. Let ν_{SAW} denote the value of ν for the ordinary self-avoiding walk. (This is expected to be the same as ν for the ordinary weakly self-avoiding walk for all $\beta > 0$.) Let ν_{MF} denote the value predicted by the mean-field approximation for the model (3). They computed that

$$\begin{aligned} \nu_{\mathrm{MF}} &= 1/2, & p &> (4-d)/2 \\ &= (2-p)/d, & p &< (4-d)/2 \end{aligned}$$

They found that the numerical results supported the conjecture that the value of ν is exactly given by $\nu = \min\{\nu_{\mathrm{SAW}}, \nu_{\mathrm{MF}}\}$.

In this paper we will study this model in one dimension. We will give nonrigorous arguments that if $p > 3/2$, then $\nu = 1/2$ and if $p < 1$, then $\nu = 1$. The results of Caracciolo *et al.* were for $2 \leqslant d \leqslant 4$, but if one simply takes $d = 1$ in their conjecture and uses $\nu_{\mathrm{SAW}} = 1$, then one obtains the same critical values of p that our nonrigorous arguments will produce. The main result of this paper is a proof that if $p < 1$ and β is sufficiently large, then the walk behaves ballistically, i.e., $\nu = 1$. More precisely, we show that there is a constant c such that for sufficiently large N, $\langle \omega(N)^2 \rangle \geqslant cN^2$. Moreover, we show that we can take the constant c as close to 1 as we like by making β sufficiently large, i.e., as $\beta \to \infty$ the speed of the walk must converge to its maximum value of one.

Brydges and Slade[3] studied a model in which the energy function equals (1) times N^{-p}. While their primary interest was in the collapse transition that occurs when $\beta < 0$, their results also cover the attractive case ($\beta > 0$). In one dimension with $p = 3/2$ they showed that the walk is diffusive for any $\beta > 0$. In two or more dimensions with $p = 1$ they showed the walk is diffusive for any $\beta > 0$. For the model studied in this paper in one dimension with $p = 3/2$, it is expected that there will be logarithmic corrections to the diffusive behavior.

Both our nonrigorous arguments and our proof make use of a real-space renormalization group transformation. In Section 2 we will define the transformation and give the heuristic arguments that the walk is ballistic for $p < 1$ and diffusive for $p > 3/2$. This argument also serves as an introduction to the proof of the main result, which appears in Section 3.

2. RENORMALIZATION GROUP TRANSFORMATION AND HEURISTIC ARGUMENTS

Given a walk ω, we "block it at scale L" as follows. The blocked walk Ω will only take values which are multiples of L. We will define times $0 = t_0 < t_1 < t_2 < \cdots < t_m \leqslant N$ for which $\Omega(i) = \omega(t_i)$. (The number of times m and hence the length of Ω will vary depending on ω.) We start by letting $t_0 = 0$. Having defined $t_0, t_1, \ldots, t_n$, we define

$$t_{n+1} = \min\{t > t_n : |\omega(t) - \omega(t_n)| = L\}$$

If the set is empty, then we take $m = n$ and we stop. In words, a single step in Ω is obtained as follows. Starting at some multiple of L, say kL, we follow ω until it has traveled a distance L either to the right or left and so it is at $(k-1)L$ or $(k+1)L$. (On the way it may return to kL, but we do not pay any attention to this.) An example of this blocking with $L = 4$ is shown in Fig. 1.

This blocking has two slightly annoying features. First, the length of Ω depends on the particular ω we start with, not just on N. The number of steps in Ω is at most N/L, but it could be as small as zero. Second, the last site visited by Ω is not necessarily the last site visited by ω. The part of ω from time t_m to N is in some sense ignored by Ω. However, during this time interval the walk ω never travels farther than $L-1$ from $\omega(t_m)$. This

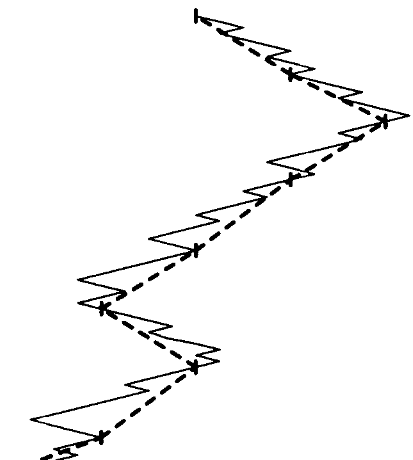

Fig. 1. An example of the RG transformation with $L = 4$. The original walk is the thin solid line, and the blocked walk is the thick dashed line. Both walks are walks in one dimension. What is shown here is a space-time plot of the walk, time being the vertical axis.

blocking can be iterated to produce walks which live on multiples of L^k for $k = 1, 2, \ldots$. Moreover, this blocking has a semigroup property. Blocking twice with scale L is the same as blocking once with scale L^2.

We now turn to the heuristic arguments that the walk is ballistic if $p < 1$ and diffusive if $p > 3/2$. It is convenient for these arguments to change the definition of the energy slightly. Instead of penalizing the walk when it visits the same site twice, we will penalize it when it traverses the same bond twice without regard to the direction in which it traverses the bond. We will also include the β in the Hamiltonian itself now. So

$$H(\omega) = \beta \sum_{s<t} \frac{1}{|t-s|^p} 1(\{\omega(s), \omega(s+1)\} = \{\omega(t), \omega(t+1)\})$$

We would like to define a new effective Hamiltonian $\tilde{H}$ for the walks Ω on scale L by

$$e^{-\tilde{H}(\Omega)} = \sum_{\omega:\omega \to \Omega} e^{-H(\omega)}$$

The notation $\omega\colon \omega \to \Omega$ means that we sum over all ω such that when ω is blocked at scale L we get Ω. The sum over ω such that $\omega \to \Omega$ may be broken up into a product over the steps in Ω of the sum over the section of ω corresponding to this step in Ω. More precisely, consider a step $\Omega(T)$, $\Omega(T+1)$ in Ω. The corresponding sequence of steps $\omega(t_T), \omega(t_T+1), \ldots,$ $\omega(t_{T+1})$ in ω satisfies

$$\begin{aligned} &\omega(t_T) = \Omega(T), \qquad \omega(t_{T+1}) = \Omega(T+1) \\ &|\omega(t) - \Omega(T)| < L \qquad \text{if} \quad t_T < t < t_{T+1} \end{aligned} \tag{4}$$

Thus we can think of the sum over $\omega\colon \omega \to \Omega$ as a product of sums, one for each step in Ω with the piece of ω in the sum subject to the constraints (4).

We now assume that β is very large. We want to compute $\tilde{H}$ to lowest order in a low-temperature expansion. For each step in Ω, the lowest-order term in the sum over the corresponding piece of ω will be the walk which goes directly from $\Omega(T)$ to $\Omega(T+1)$ in L steps. So to lowest order, $\tilde{H}(\Omega) = H(\omega_0)$, where ω_0 is the walk on the unit lattice which follows Ω exactly and so has L steps for each step in Ω.

Let T and S be such that the bonds $\{\Omega(T), \Omega(T+1)\}$ and $\{\Omega(S), \Omega(S+1)\}$ are the same. Then there will be L pairs of bonds from the corresponding sections of ω that are the same. They each receive an energy penalty of $\beta/|t-s|^p$. Since one step in Ω corresponds to L steps in ω, we have $|t-s| \approx L\,|T-S|$. So the energy in $\tilde{H}(\Omega)$ associated with the bonds $\{\Omega(T), \Omega(T+1)\}$ and $\{\Omega(S), \Omega(S+1)\}$ is $\beta L/(|T-S|\, L)^p$. Thus to lowest order, $\tilde{H}$ is given by

$$\tilde{H}(\Omega) = \beta L^{1-p} \sum_{S<T} \frac{1}{|T-S|^p} 1(\{\Omega(S), \Omega(S+1)\} = \{\Omega(T), \Omega(T+1)\})$$

So $\tilde{H}$ has the same form as H with β replaced by βL^{1-p}. If $p<1$, the renormalized β is larger and if $p>1$, it is smaller. Thus the $\beta=\infty$ fixed point of the renormalization group transformation is stable. This suggests that the walk will behave ballistically when $p<1$, at least for large β.

We now turn to the heuristic argument that when $p>3/2$ the walk behaves diffusively, i.e., $v=1/2$. For this argument we assume that β is small and compute the leading term in a high-temperature expansion. As before, we think of the sum over ω such that $\omega \to \Omega$ as a product of sums, one for each step in Ω. For a step $\Omega(T)$, $\Omega(T+1)$ in Ω we sum over a piece of ω which goes from $\Omega(T)$ to $\Omega(T+1)$ subject to the constraints (4). In the low-temperature expansion there was a single dominant term in this expansion. When β is small, the sum will not be dominated by a single term. The walk ω will wander from $\Omega(T)$ to $\Omega(T+1)$ in $O(L^2)$ steps. Consequently it will visit each of the L bonds between $\Omega(T)$ and $\Omega(T+1)$ $O(L)$ times. If $\{\Omega(S), \Omega(S+1)\} = \{\Omega(T), \Omega(T+1)\}$, then the section of ω corresponding to the step $\Omega(S)$, $\Omega(S+1)$ will also visit each of these L bonds $O(L)$ times. So the total number of pairs of bonds in ω that are equal and correspond to these two steps in Ω will be $LO(L^2) = O(L^3)$.

When β is small, each step in Ω corresponds to $O(L^2)$ steps in ω. Hence the number of steps $|t-s|$ in ω between the visit to the same bond is related to the number of steps $|T-S|$ in Ω between the visit to the same bond on scale L by $|t-s| = O(L^2)\,|T-S|$. Thus the energy in $\tilde{H}$ corresponding to the two steps $\Omega(S)$, $\Omega(S+1)$ and $\Omega(T)$, $\Omega(T+1)$ is $\beta O(L^3)/(|T-S|\, L^2)^p = \beta O(L^{3-2p})/|T-S|^p$. The new Hamiltonian is again of the same form with β replaced by $\beta c L^{3-2p}$, where c is a constant. If $p>3/2$, then we can use an L large enough that cL^{3-2p} is less than 1 and conclude that β decreases. So the $\beta=0$ fixed point of the renormalization group is stable. This suggests that the walk behaves diffusively when $p>3/2$.

3. PROOF IN THE BALLISTIC REGIME

We now state and prove our main result. In the following theorem the energy function is given by (3).

Theorem 1. Let $p<1$. For every $\alpha<1$ there is a β_0 such that $\beta \geqslant \beta_0$ implies

$$\lim_{N \to \infty} \operatorname{Prob}(|\omega(N)| \geqslant \alpha N) = 1$$

We denote ω blocked at scale L^k by $\Omega_k(\omega)$, i.e., $\Omega_k(\omega)$ is ω after k applications of the renormalization group transformation. If Ω is a walk on some scale, then $|\Omega|$ will denote the number of steps in Ω. For example, $|\Omega_k(\omega)|$ is the number of steps in $\Omega_k(\omega)$, each step being of length L^k. Let Ω be a walk at some scale L^k. We define $R(\Omega)$ to be the number of reversals in Ω, i.e., the number of t such that $\Omega(t-1)=\Omega(t+1)$.

Define $\delta=(1-p)/3$ and

$$\gamma_k = L^{-1-\delta(k+1)}$$

Then define γ by

$$1-\gamma = \prod_{k=0}^{\infty} (1-3L\gamma_k)$$

The constants γ_k converge to zero fast enough that the product converges, and we can make γ as close to 1 as we like by making L large enough.

Now we define the exceptional events.

$$E_k = \{\omega\colon R(\Omega_k(\omega)) > \gamma_k \, |\Omega_k(\omega)|\}$$

Here k starts at 0 and $\Omega_0(\omega)$ is just ω. The ratio $R(\Omega_k(\omega))/|\Omega_k(\omega)|$ is the fraction of steps in Ω blocked at scale L^k for which the walk changes direction. Thus the exceptional event contains those walks for which this fraction is bigger than γ_k. Let n be the integer for which $L^n \leqslant N < L^{n+1}$. There are two main parts to the proof.

1. Show that walks which do not belong to any of the E_k are ballistic.

2. Show that the probability of the union of the E_k is small.

In a full-fledged renormalization group argument we would compute a new effective Hamiltonian after one application of the renormalization group transformation. We would then rescale to get back to the unit lattice and attempt to iterate this process. While our proof uses renormalization group ideas, we do not attempt to compute a new effective Hamiltonian and we do not rescale. The renormalization group transformation is used to define the "bad" events. But these events are always subsets of the original set of walks on the unit lattice. We begin with step 1, which is a purely deterministic statement.

Lemma 2. For every walk ω,

$$|\Omega_{k+1}(\omega)| + 1 \geqslant \frac{1}{L} |\Omega_k(\omega)| - 2R(\Omega_k(\omega))$$

Proof. Consider a single step in $\Omega_{k+1}(\omega)$. Let m be the number of steps in the corresponding section of $\Omega_k(\omega)$ and let r be the number of reversals which take place in this corresponding section. If $r=0$, then m is exactly L. Each reversal allows $\Omega_k(\omega)$ to spend at most an additional $2L$ steps in traversing the step in $\Omega_{k+1}(\omega)$. (This is a slightly less than optimal bound.) Hence $m \leqslant (2r+1)L$. Rewriting this bound as $1 \geqslant m/L - 2r$ and summing over the bonds in $\Omega_{k+1}(\omega)$ yields the lemma. The section of $\Omega_k(\omega)$ at the very end which does not correspond to a step in $\Omega_{k+1}(\omega)$ contributes the $+1$ to the left side of the inequality in the lemma. ■

Proposition 3. If $\omega \notin E_k$ for $k = 0, 1, 2,..., n-1$, then $|\omega(N)| \geqslant (1-\gamma-1/L)N$, where $1-\gamma = \prod_{i=0}^{\infty} (1-3L\gamma_i)$.

Proof. If $\omega \notin E_k$, then the lemma implies

$$\begin{aligned} |\Omega_{k+1}(\omega)| &\geqslant (1-2L\gamma_k)\, |\Omega_k(\omega)|/L - 1 \\ &= (1-3L\gamma_k)\, |\Omega_k(\omega)|/L + \gamma_k\, |\Omega_k(\omega)| - 1 \end{aligned}$$

Both γ_k and $|\Omega_k(\omega)|$ decrease as k increases. Let l be the last integer such that $\gamma_l\, |\Omega_l(\omega)| \geqslant 1$. [It is easy to see from the definition of γ_k and the trivial bound $|\Omega_k(\omega)| \leqslant N/L^k$ that $\gamma_k\, |\Omega_k(\omega)|$ becomes less than 1 long before k reaches n.] For $k \leqslant l$ we have

$$|\Omega_{k+1}(\omega)| \geqslant (1-3L\gamma_k)\, |\Omega_k(\omega)|/L$$

Thus

$$|\Omega_{l+1}(\omega)| \geqslant L^{-(l+1)} \prod_{i=0}^{l} (1-3L\gamma_i)\, N \geqslant (1-\gamma)\, NL^{-(l+1)}$$

where $1-\gamma = \prod_{i=0}^{\infty} (1-3L\gamma_i)$.

Since $\gamma_{l+1}\, |\Omega_{l+1}(\omega)| < 1$ and $\omega \notin E_{l+1}$, we have $R(\Omega_{l+1}(\omega)) < 1$ and so $R(\Omega_{l+1}(\omega))$ must be zero. Thus $\Omega_{l+1}(\omega)$ consist of at least $(1-\gamma)\, NL^{-(l+1)}$ steps of size L^{l+1} in the same direction. The final partial step may wipe out one of them, so we have

$$\begin{aligned} |\omega(N)| &\geqslant [(1-\gamma)\, NL^{-(l+1)} - 1]\, L^{l+1} \\ &= (1-\gamma-L^{l+1}/N)\, N \geqslant (1-\gamma-1/L)\, N \end{aligned}$$

The last inequality uses $N \geqslant L^n$ and the fact that l is much less than n. ■

Now we turn to showing that the probability of E_k is small. Actually, we will show that the probability of $E_k \backslash \bigcup_{i=0}^{k-1} E_i$ is small. Fix a value of k. Most of the following definitions will depend on k, but we will not make this dependence explicit, to keep the notation from becoming too cumbersome. We will break E_k up into a bunch of smaller sets which we now define.

Let $r = R(\Omega_k(\omega))$ be the number of reversals in the walk ω blocked at scale L^k. For convenience let Ω denote $\Omega_k(\omega)$. Let i_j, $j = 1, 2,..., r$, be the times in Ω at which the reversals occur, i.e., $\Omega(i_j - 1) = \Omega(i_j + 1)$ for $j = 1, 2,..., r$. These reversals come in two types. The walk can be going to the right and then switch to the left. We call this an RL reversal. In this case $\Omega(i_j - 1) = \Omega(i_j + 1) = \Omega(i_j) - 1$. When the walk switches from left to right we have $\Omega(i_j - 1) = \Omega(i_j + 1) = \Omega(i_j) + 1$, and we refer to this as an LR reversal.

Let t_i be the time in ω which corresponds to time i in Ω. So the section of ω for $t = t_i$ to $t = t_{i+1}$ corresponds to the single step of size L^k in Ω for times i, $i+1$. For the jth reversal we let l_j be distance that ω overshoots $\Omega_k(\omega)$ at the jth reversal (see Fig. 2). More precisely, for an RL reversal

$$l_j = \max_{t_{i_j} \leqslant t \leqslant t_{i_j} + 1} \omega(t) - \omega(t_{i_j})$$

For an LR reversal

$$l_j = \max_{t_{i_j} \leqslant t \leqslant t_{i_j} + 1} -\omega(t) + \omega(t_{i_j})$$

Of course $0 \leqslant l_j < L^k$. We denote the sequence $(l_j)_{j=1}^r$ by $\underline{l}(\omega)$.

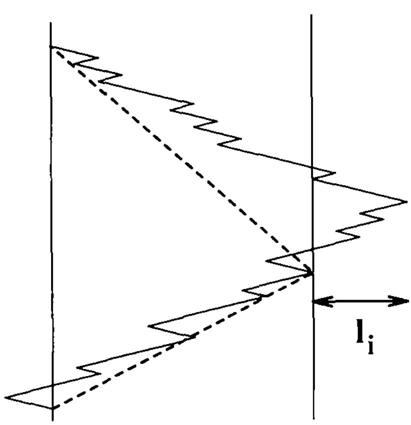

Fig. 2. An illustration of the definition of l_i.

Now fix an Ω at scale L^k, let $r = R(\Omega)$, and fix an $\underline{l}$. Define

$$E(\Omega, \underline{l}) = \{\omega: \Omega_k(\omega) = \Omega,\ \underline{l}(\omega) = \underline{l}\}$$

Suppose $\omega \notin E_i$ for $i < k$ and $\omega \in E_k$. We argued before that $|\Omega_k(\omega)| \geqslant (1-\gamma)\, N/L^k$. [Actually, we only proved this bound for $k \leqslant l+1$, where l is the largest integer with $\gamma_l |\Omega_l(\omega)| \geqslant 1$. If $\omega \in E_k$, then $\Omega_k(\omega)$ has at least one reversal in it. Thus $\Omega_{k-1}(\omega)$ must have at least one reversal. Since $\omega \notin E_{k-1}$, this implies $\gamma_{k-1}\, |\Omega_{k-1}(\omega)| \geqslant 1$. Now l is the largest integer such that $\gamma_l\, |\Omega_l(\omega)| \geqslant 1$, so we must have $k-1 \leqslant l$.] Thus $\omega \in E_k \backslash \bigcup_{i=0}^{k-1} E_i$ implies

$$R(\Omega_k(\omega)) > \gamma_k (1-\gamma)\, N/L^k \tag{5}$$

Hence

$$E_k \Big\backslash \bigcup_{i=0}^{k-1} E_i \subset \bigcup_{\Omega} \bigcup_{\underline{l}} E(\Omega, \underline{l}) \tag{6}$$

where the union over Ω is only over Ω with $R(\Omega) > \gamma_k(1-\gamma)\, N/L^k$.

Proposition 4. There exists an $\varepsilon' > 0$ (depending on L) such that for $k = 0, 1, \ldots, n-1$

$$\operatorname{Prob}\left(E_k \Big\backslash \bigcup_{i=0}^{k-1} E_i\right) \leqslant 2 \exp(-\beta \varepsilon' L^{n\delta})$$

[Recall that $\delta = (1-p)/3$.]

Before we prove Proposition 4, we complete the proof of Theorem 1.

Proof of Theorem 1. Note that it is $L^n\delta$ that appears in the right side of Proposition 4, not $L^k\delta$. Hence Proposition 4 implies

$$\operatorname{Prob}\left(\bigcup_{k=0}^{n-1} E_k\right) \leqslant 2n \exp(-\beta \varepsilon' L^{n\delta})$$

By Proposition 3,

$$\{\omega: |\omega(N)| < (1-\gamma-1/L)\, N\} \subset \bigcup_{k=0}^{n-1} E_k$$

We choose L large enough that $(1-\gamma-1/L) \geqslant \alpha$. (Recall that we can make γ as small as we want by making L sufficiently large.) The theorem now follows since $n \to \infty$ as $N \to \infty$. ■

We will use the following two lemmas to prove Proposition 4. The proof of the second lemma is trivial and will not be given.

Lemma 5. There is a constant $\varepsilon > 0$ which depends on L such that for a walk Ω at scale L^k

$$\mathrm{Prob}(E(\Omega, \underline{l})) \leqslant \exp[-\beta\varepsilon L^{2k\delta} R(\Omega)]$$

Lemma 6. For every $r > 0$ there is a constant $c(r)$ such that $c \geqslant c(r)$ implies $xe^{-cx^r} \leqslant 1$ for all $x \geqslant 0$.

Proof of Proposition 4. At each reversal the number of possibilities for l_i is L^k since $0 \leqslant l_i < L^k$. Thus, given Ω, the number of possible $\underline{l}$ is $L^{kR(\Omega)}$. So the inclusion (6) and Lemma 5 imply

$$\mathrm{Prob}\left(E_k \Big\backslash \bigcup_{i=0}^{k-1} E_i\right) \leqslant \sum_{\Omega:\, R(\Omega) > \gamma_k(1-\gamma) N/L^k} L^{kR(\Omega)} \exp[-\beta\varepsilon L^{2k\delta} R(\Omega)]$$

Using Lemma 6 and assuming β is large enough,

$$L^k \exp[-\tfrac{1}{2}\beta\varepsilon L^{2k\delta}] \leqslant 1$$

So

$$\begin{aligned}\mathrm{Prob}\left(E_k \Big\backslash \bigcup_{i=0}^{k-1} E_i\right) &\leqslant \sum_{\Omega:\, R(\Omega) > \gamma_k(1-\gamma) N/L^k} \exp[-\tfrac{1}{2}\beta\varepsilon L^{2k\delta} R(\Omega)] \\ &\leqslant \exp[-\tfrac{1}{2}\beta\varepsilon L^{2k\delta}\gamma_k(1-\gamma)\, N/L^k]\, M\end{aligned}$$

where M is the number of possible Ω. The number of steps in Ω is at most N/L^k, and the number of walks with $\leqslant N/L^k$ steps is $2^{1+N/L^k}$, so the probability is now

$$\leqslant 2\{2\exp[-\tfrac{1}{2}\beta\varepsilon L^{2k\delta}\gamma_k(1-\gamma)]\}^{N/L^k}$$

Using the definition of γ_k, this becomes

$$= 2\{2\exp[-\tfrac{1}{2}\beta\varepsilon L^{k\delta-1-\delta}(1-\gamma)]\}^{N/L^k}$$

If β is sufficiently large we can pick $\varepsilon' > 0$ so that

$$2\exp[-\tfrac{1}{2}\beta\varepsilon L^{k\delta-1-\delta}(1-\gamma)] \leqslant \exp(-\beta\varepsilon' L^{k\delta})$$

Recalling that $N \geqslant L^n$, we see that the probability is at most

$$\leqslant 2\exp(-\beta\varepsilon' L^{k\delta} N/L^k) \leqslant 2\exp(-\beta\varepsilon' L^{k\delta+n-k}) \leqslant 2\exp(-\beta\varepsilon' L^{n\delta}) \quad \blacksquare$$

Proof of Lemma 5. We define

$$Z(\Omega, \underline{l}) = \sum_{\omega \in E(\Omega, \underline{l})} e^{-\beta H(\omega)}$$

so that

$$\mathrm{Prob}(E(\Omega, \underline{l})) = Z(\Omega, \underline{l})/Z$$

For each Ω and $\underline{l}$ we will define a map on walks, $\omega \to \bar{\omega}$, which can be thought of as "unfolding" the walk in a way that depends on Ω and $\underline{l}$. If all the l_i were 0, this map would be given as follows. We can unfold the walk Ω to produce a walk with the same number of steps of length L^k, but with all the steps going to the right. We do the same transformation to ω. Each section of ω which corresponds to a single step in Ω is left rigid. This section is translated and possibly flipped in the same way that the corresponding step in Ω is translated and possibly flipped. When all the $l_i = 0$, it is easy to see that the new walk $\bar{\omega}$ has energy less than or equal to the energy of the original walk. However, if some of the l_i are not zero, this need not be the case. In fact the energy of $\bar{\omega}$ can be much higher. See Fig. 3 for an example. To avoid this increase in energy we must modify the definition of the unfolding.

As before, t_i denotes the time in ω which corresponds to the time i in Ω. In particular, $\omega(t_i) = \Omega(i)$. Also, Ω has reversals at the times i_j for $j = 1, 2,..., r$. For each reversal we define another time s_j which we will refer to as a "flip time." The time s_j will lie between t_{i_j} and t_{i_j+1}. Consider an RL reversal which occurs at time j in Ω. Between $t = t_{i_j}$ and $t = t_{i_j+1}$ the walk

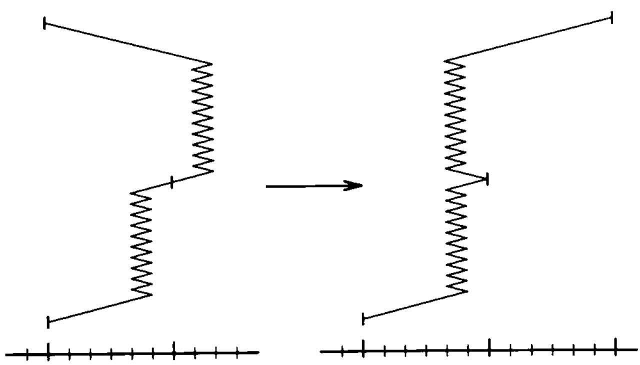

Fig. 3. An example of a walk for which the naive unfolding greatly increases the energy rather than lowering it as needed for the Peierls argument.

ω travels from $\Omega(i_j)$ to $\Omega(i_j+1)=\Omega(i_j)-L^k$. At some time (or possibly several times) it hits the site $\Omega(i_j)+l_j$. The time s_j is the first such time, i.e.,

$$s_j = \min\{t \geqslant t_{i_j} : \omega(t) = \Omega(i_j) + l_j\}$$

For LR reversals

$$s_j = \min\{t \geqslant t_{i_j} : \omega(t) = \Omega(i_j) - l_j\}$$

We will write down a formula for $\bar{\omega}$ in a moment, but a verbal description may be more informative. We have defined the flip times s_j for $j=1, 2,..., r$. Let $s_0=0$ and $s_{r+1}=N$. In the unfolding $\omega \to \bar{\omega}$, we keep the walk rigid on each time interval $[s_j, s_{j+1}]$, $j=0, 1,..., r$. We allow the walk to pivot at the times s_j and then unfold the walk to produce a walk $\bar{\omega}$ that is as long as possible given the rigidity constraints. Figure 4 shows the unfolding process for a single reversal. The formula for $\bar{\omega}$ is as follows. Let $s_j \leqslant t \leqslant s_{j+1}$. If j is odd,

$$\bar{\omega}(t) = \bar{\omega}(s_j) + [\omega(s_j) - \omega(t)]$$

If j is even,

$$\bar{\omega}(t) = \bar{\omega}(s_j) + [\omega(t) - \omega(s_j)]$$

This unfolding has a property that will be crucial for our energy estimates. For $s_j \leqslant t \leqslant s_{j+1}$, $\omega(t)$ stays between $\omega(s_j)$ and $\omega(s_{j+1})$. Thus $\bar{\omega}(t)$ stays between $\bar{\omega}(s_j)$ and $\bar{\omega}(s_{j+1})$. If the first step in Ω is to the right, then the unfolding process results in $\bar{\omega}(s_0) < \bar{\omega}(s_1) < \cdots < \bar{\omega}(s_{n+1})$. If the first step is to the left, then we have $\bar{\omega}(s_0) > \bar{\omega}(s_1) > \cdots > \bar{\omega}(s_{n+1})$. Thus the energy of $\bar{\omega}$ does not contain any terms with s and t coming from different $[s_j, s_{j+1}]$ intervals. For s and t which come from the same $[s_j, s_{j+1}]$

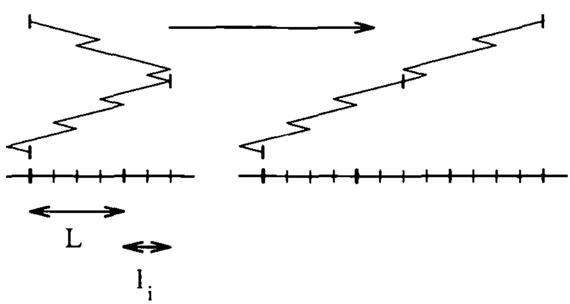

Fig. 4. The unfolding process that we use in the Peierls argument, for a single reversal in the blocked walk.

interval, we see that $\bar{\omega}(s)=\bar{\omega}(t)$ if and only if $\omega(s)=\omega(t)$, since ω is left rigid on $[s_j, s_{j+1}]$. Thus $H(\bar{\omega})\leqslant H(\omega)$.

We have shown that the unfolding does not raise the energy of the walk, but we want to show that in fact it lowers the energy by at least an amount proportional to the number of reversals in Ω. Consider an RL reversal. (The argument for the LR case is essentially the same.) The walk ω is at the same site at times t_{i_j-1} and t_{i_j+1}. In between these times we have the flip time s_j. At this time the walk is at least a distance L^k from its position at t_{i_j-1} and t_{i_j+1}. Thus there are at least L^k pairs s, t with $\omega(s)=\omega(t)$, $t_{i_j-1}\leqslant s<s_j$ and $s_j<t\leqslant t_{i_j+1}$. Define $T_j=t_{i_j+1}-t_{i_j-1}$. Then the above times s, t satisfy $1/|s-t|^p\geqslant 1/T_j^p$. So we have proved that

$$H(\omega)\geqslant H(\bar{\omega})+L^k\sum_{j=1}^{R(\Omega)} T_j^{-p} \tag{7}$$

The proof is completed by the following two lemmas.

Lemma 7. There is an $\varepsilon>0$ such that

$$\sum_{i=1}^{R(\Omega)} T_i^{-p}\geqslant \varepsilon L^{-k(1-2\delta)}R(\Omega)$$

Lemma 8. The map $\omega\to\bar{\omega}$ is one to one on $E(\Omega, \underline{l})$.

To complete the proof of Lemma 5, Eq. (7) and Lemma 7 imply

$$Z(\Omega, \underline{l})\leqslant \sum_{\omega\in E(\Omega,\underline{l})} \exp[-\beta H(\bar{\omega})-\beta\varepsilon L^{2k\delta}R(\Omega)]\leqslant \exp[-\beta\varepsilon L^{2k\delta}R(\Omega)]\, Z$$

where the last inequality uses Lemmas 8. ■

Proof of Lemma 7. Each step in the original walk can take part in at most two reversals in $\Omega_k(\omega)$. Thus $\sum_j T_j\leqslant 2N$. So at least $\frac{1}{2}R(\Omega)$ of the T_j satisfy $T_j\leqslant 4N/R(\Omega)$. Using Eq. (5), we have

$$\frac{4N}{R(\Omega)}\leqslant\frac{4N}{\gamma_k(1-\gamma)\,NL^{-k}}=\frac{4L^k}{\gamma_k(1-\gamma)}=\frac{4}{1-\gamma}\,L^{(k+1)(1+\delta)}$$

Thus

$$\sum_j T_j^{-p}\geqslant\frac{1}{2}R(\Omega)\left[\frac{4}{(1-\gamma)}\right]^{-p} L^{-(k+1)p(1+\delta)}$$

Using

$$p(1+\delta)=(1-3\delta)(1+\delta)=1-2\delta-3\delta^2\leqslant 1-2\delta$$

we have

$$L^{-(k+1)p(1+\delta)} \geqslant L^{-(k+1)(1-2\delta)}$$

and the lemma follows. ■

Proof of Lemma 8. We show the map is one to one by showing that given Ω and $\underline{l}$, one can reconstruct ω from $\bar{\omega}$. Obviously, if we knew the flip times s_j we could fold $\bar{\omega}$ to get ω. We have

$$\bar{\omega}(s_j) = i_j L^k + l_j + 2 \sum_{m=1}^{j-1} l_m \tag{8}$$

Thus, given Ω and $\underline{l}$, we can find the sites $\bar{\omega}(s_j)$. There may be several times at which $\bar{\omega}$ hits these sites, but from the definition of s_j we see that s_j is the first time for which Eq. (8) holds. Thus, given $\bar{\omega}$, Ω, and $\underline{l}$, we can find the times s_j, and so we can fold $\bar{\omega}$ to reconstruct ω. ■

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